# Nonlinear Evolution of Internal Ideal MHD Modes Near the Boundary of Marginal Stability

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*To Professor Arnulf Schlüter on his 60th Birthday*

A family of ideal MHD equilibria is considered introducing the concept of a driving parameter $\lambda$ the increase of which beyond a certain threshold $\lambda_0$ drives the plasma from a linearly stable to an unstable state. Using reductive perturbation theory, the nonlinear ideal MHD equations of motion are expanded in the neighbourhood of $\lambda_0$ with respect to a small parameter $\varepsilon$. An appropriate scaling for the expansions is derived from the linear eigenmode problem. Integrability conditions for the reduced nonlinear equations yield nonlinear amplitude equations for the marginal mode. Nonlinearly, the instabilities are either oscillations about bifurcating equilibria, or they are explosive. In the latter case, the stability limit depends on the amplitude of the perturbation and is shifted into the linearly stable regime. Generally bifurcation of dynamically connected equilibria is observed at $\lambda_0$.

## 1. Introduction

Among macroscopic large-scale instabilities the so called ideal MHD instabilities are of major importance due to their potential destructiveness. Their essential properties are qualitatively unaffected by nonideal effects like resistivity, viscosity, anisotropy, etc. so that they may be rather well described by using the framework of ideal MHD theory. Ideal MHD instabilities do certainly belong to the most dangerous plasma instabilities since they may destroy an equilibrium structure within a fraction of the confinement time admitted by nonideal effects. Thus, stability of thermonuclear plasmas with respect to ideal MHD modes is an indispensable requirement.

The linear theory of ideal MHD modes is already a rather complex topic, and over the last two decades great effort has been devoted to its investigation. A survey of this field and lists of representative references may be found e.g. in Refs. [1] and [2]. At least qualitatively, linear stability is now fairly wellunderstood — last but not least by extensive use of modern computers. Since a better understanding of linear problems was urgent and difficult, the interest was for a long time mainly concentrated on this theory.

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Experimentally, no plasma is completely stable, but it has been observed that, due to nonlinear saturation effects, one can live with certain instabilities. In the endeavour to understand those effects as well as the physical nature of certain dramatic instabilities like the disruptive instability, interest arose for studying the structure of nonlinear instabilities and the nonlinear wave-wave interaction. In the case of specific modes nonlinear theories were developed [3, 4, 5]. More general situations were studied by computer simulation [6]. Searching for stable ideal MHD equilibria by numerical minimization of the plasma energy still another approach to the problem of nonlinear stability appeared [7, 8].

In this paper, the nonlinear evolution of ideal MHD modes near the boundary of linear stability is studied. We assume the existence of a „driving parameter" $\lambda$ whose increase beyond a threshold $\lambda_0$ leads a plasma equilibrium from a linearly stable to an unstable state. Introducing a small parameter $\varepsilon$ so that $\varepsilon \to 0$ when $\lambda \to \lambda_0$, we treat the nonlinear ideal MHD equations of motion by using a reductive perturbation method [9]. This method has its origin in bifurcation theory and was developed by Andronow, Hopf, Bogoljubow, Mitropolski and others. It was successfully applied in hydrodynamics, see e.g. Ref. [10], and physical chemistry [11] in order to study the formation of dissipative structures. There, a close connection to equilibrium bifurcation was found. Bifurcation of equilibria has also been studied in MHD theory [12], taking into account the top-

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ological constraints imposed by the dynamics of the problem only to lowest order. Since in ideal plasmas, the magnetic field lines are frozen into the fluid, the field line topology must be left unaltered by bifurcation in order that the new equilibria be dynamically accessible from the original one. This dynamical connection constraint provides a severe complication of the bifurcation problem in the sense that studying nonlinear motion reveals a connection to equilibrium bifurcation but not vice versa.

The paper is organized in the following way: In Sect. 2 we introduce the concept of the driving parameter $\lambda$ into the theory of ideal MHD-equilibria and expand the equilibrium equations in powers of $\lambda - \lambda_0$ where $\lambda_0$ determines the linear stability limit. In Sect. 3 we derive a scaling for the expansion of nonequilibrium quantities by considering almost marginal eigenmodes of the linearized equations of motion. In Sect. 4 the nonlinear relation between the fluid displacement and the velocity $\boldsymbol{v}$ will be resolved. In Sect. 5 we reduce the nonlinear equations of motion by introducting the perturbation expansions obtained in Sect. 3 (reductive perturbation method). In Sects. 6 and 7 we derive nonlinear amplitude equations for the marginal mode. Finally, in Sect. 8, the different types of motion allowed by the theory will be studied. Bifurcation of dynamically connected equilibria will be found and related with the nonlinear motions. In the Appendix a list of all operators which are used in the paper is given in order to enable a more compact mathematical representation.

This paper restricts to the case of internal modes, i.e. the position of the plasma boundary is unaffected by the plasma motion. Internal modes may appear naturally, or they may be forced by the assumption that the plasma is in direct contact with a wall.

## 2. Equilibrium and Driving Parameter λ

Let us consider a family of ideal MHD equilibria which depends on one or several parameters $\lambda_1, \lambda_2, \ldots$. The limit between linearly stable and unstable will generally depend on these parameters and will be represented by a surface in parameter space (Figure 1). Parameters without influence on stability will not be taken into account. Parameters the range of which extends across the stability boundary are called "driving parameters". We shall

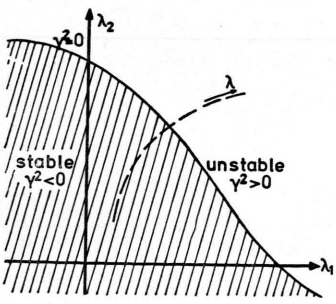


Fig. 1. Boundary of linear stability in parameter space.

only consider equilibria for which at least one driving parameter exists.

If there are several driving parameters, it will always be possible to find a path $\lambda_1(\lambda)$, $\lambda_2(\lambda)$, ... (dotted curve in Fig. 1) which leads from the stable into the unstable region. Thus, the transition from linearly stable to unstable may be described by a single parameter. Therefore we shall consider in this paper equilibria with only one driving parameter[1]. According to this definition the linear growth rates depend on $\lambda$, and we complete our definition by the requirement

$$\gamma^2 = 0\,, \quad \mathrm{d}\gamma^2/\mathrm{d}\lambda > 0 \quad \text{for} \quad \lambda = \lambda_0\,, \tag{1}$$

where $\gamma$ denotes the linear growth rate of the most unstable eigenmode. Further we shall assume in this paper that $\gamma$ is an isolated eigenvalue of the linear stability operator.

There are many MHD equilibria for which a driving parameter exists, and usually there exist even many different possibilities of choosing $\lambda$. Specific examples of driving parameters are $\beta = 2p/B^2$, geometrical parameters determining the plasma shape, parameters determining the pressure profile or, in tokamaks, the inverse safety factor. Applications of the theory presented in this paper may be appreciably simplified by properly choosing the driving parameter.

All the considerations in this paper will be restricted to $\lambda$ values in the neighbourhood of the critical value $\lambda_0$. We assume that the equilibrium values $\boldsymbol{B}_0(\boldsymbol{r}, \lambda)$ and $p_0(\boldsymbol{r}, \lambda)$ of the magnetic field

[1] For studying specific problems it might sometimes be useful to keep the explicit dependence on more than one parameter.

and pressure can be expanded in power series

$$\begin{aligned} \boldsymbol{B}_0 &= \boldsymbol{B}_{00} + \tau \boldsymbol{B}_{01} + \tau^2 \boldsymbol{B}_{02} + \tau^3 \boldsymbol{B}_{03} + \cdots, \\ p_0 &= p_{00} + \tau p_{01} + \tau^2 p_{02} + \tau^3 p_{03} + \cdots, \end{aligned} \tag{2}$$

where

$$\tau = \lambda - \lambda_0, \tag{3}$$

and where $p_{00} = p_0(\boldsymbol{r}, \lambda_0)$, $p_{01} = \partial p_0(\boldsymbol{r}, \lambda_0)/\partial\lambda_0$ etc. We shall assume that $\varrho_0$ does not depend on $\lambda$ and write

$$\varrho_0 = \varrho_{00}. \tag{4}$$

For the sake of simplicity, we assume

$$\nabla \varrho_{00} \equiv 0. \tag{5}$$

Static ideal MHD equilibria have to satisfy the Eqs.

$$\boldsymbol{j}_0 \times \boldsymbol{B}_0 = \nabla p_0, \quad \boldsymbol{\nabla} \cdot \boldsymbol{B}_0 = 0, \tag{6}$$

where

$$\boldsymbol{j} = \boldsymbol{\nabla} \times \boldsymbol{B}. \tag{7}$$

Introducing the expansions (2) into the equilibrium Eqs. (6) and equating the coefficients of equal powers in $\tau$, we obtain the Eqs.

$$\begin{aligned} \boldsymbol{j}_{00} \times \boldsymbol{B}_{00} &= \nabla p_{00}, \\ \boldsymbol{j}_{00} \times \boldsymbol{B}_{01} + \boldsymbol{j}_{01} \times \boldsymbol{B}_{00} &= \nabla p_{01}, \\ \boldsymbol{j}_{00} \times \boldsymbol{B}_{02} + \boldsymbol{j}_{02} \times \boldsymbol{B}_{00} + \boldsymbol{j}_{01} \times \boldsymbol{B}_{01} &= \nabla p_{02}, \\ \boldsymbol{j}_{00} \times \boldsymbol{B}_{03} + \boldsymbol{j}_{01} \times \boldsymbol{B}_{02} & \\ + \boldsymbol{j}_{02} \times \boldsymbol{B}_{01} + \boldsymbol{j}_{03} \times \boldsymbol{B}_{00} &= \nabla p_{03}, \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ \boldsymbol{\nabla} \cdot \boldsymbol{B}_{0i} = 0, \quad i &= 0, 1, 2, \ldots . \end{aligned} \tag{8}$$

### 3. Scaling of Timedependent Perturbations

In order to find an appropriate expansion of the timedependent equilibrium perturbations, we shall first consider the linear eigenmode problem (see, e.g. [13])

$$\varrho_{00}\, \partial^2 \boldsymbol{\xi}/\partial t^2 = \varrho_{00}\, \gamma^2 \boldsymbol{\xi} = \mathbf{F}_0(\boldsymbol{\xi}), \tag{9}$$

$\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{r}, t)$ being the linear displacement of a fluid element and $\mathbf{F}_0$ the linear MHD stability operator for the equilibrium $\boldsymbol{B}_0$, $p_0$ (Eq. (A7) in the Appendix).

We expand Eq. (9) about $\lambda = \lambda_0$ setting

$$\tau = \tau_1 \varepsilon + \tau_2 \varepsilon^2 + \tau_3 \varepsilon^3 + \cdots \tag{10}$$

and

$$\boldsymbol{\xi} = \varepsilon \boldsymbol{\xi}_1 + \varepsilon^2 \boldsymbol{\xi}_2 + \cdots, \tag{11}$$

where $\varepsilon$ is a small expansion parameter. From the expansions (2) we get

$$\mathbf{F}_0 = \mathbf{F}_{00} + \tau \mathbf{F}_{01} + \tau^2 \mathbf{F}_{02} + \cdots, \tag{12}$$

the operators $\mathbf{F}_{00}$, $\mathbf{F}_{01}$ and $\mathbf{F}_{02}$ being defined in the Appendix. Inserting Eqs. (11) and (12) in Eq. (9) yields

$$\begin{aligned} &\varrho_{00}\, \gamma^2 \boldsymbol{\xi}_1 + \cdots \\ &= \mathbf{F}_{00}(\boldsymbol{\xi}_1) + \varepsilon \mathbf{F}_{00}(\boldsymbol{\xi}_2) + \tau \mathbf{F}_{01}(\boldsymbol{\xi}_1) + \cdots . \end{aligned} \tag{13}$$

According to Eqs. (1), (3) and (10) we have $\gamma^2 = 0$ for $\varepsilon = 0$, and thus, from Eq. (13) we get to lowest order

$$0 = \mathbf{F}_{00}(\boldsymbol{\xi}_1), \tag{14}$$

i.e. $\boldsymbol{\xi}_1$ is the marginal eigenmode. Since for internal modes the operator $\mathbf{F}_{00}$ is selfadjoint [13], using notation (A2) we have

$$(\boldsymbol{\xi}_1, \mathbf{F}_{00}(\boldsymbol{\xi}_2)) = (\boldsymbol{\xi}_2, \mathbf{F}_{00}(\boldsymbol{\xi}_1)) = 0. \tag{15}$$

Eqs. (13)—(15) lead to

$$\gamma^2 = \tau (\boldsymbol{\xi}_1, \mathbf{F}_{01}(\boldsymbol{\xi}_1))/(\boldsymbol{\xi}_1, \varrho_{00} \boldsymbol{\xi}_1) + \cdots .$$

Thus,

$$\partial \boldsymbol{\xi}_1/\partial t = \gamma \boldsymbol{\xi}_1 + \cdots \sim \sqrt{\tau}\, \boldsymbol{\xi}_1,$$

i.e. time derivatives of first order scale like $\sqrt{\tau}$ as $\tau \to 0$. This result suggests the introduction of a slow time scale $T$ defined by

$$\partial/\partial t = \sqrt{\tau}\, \partial/\partial T \tag{16}$$

with the consequence that

$$\partial \boldsymbol{\xi}_1/\partial T = \gamma_{\mathrm{T}} \boldsymbol{\xi}_1, \quad \gamma_{\mathrm{T}} = \gamma/\sqrt{\tau} = O(1).$$

Note that, due to the appearence of $\sqrt{\tau}$, our investigation is for the time being restricted to linearly unstable modes ($\tau > 0$). Linearly stable modes ($\tau < 0$) will be included through a slight modification in Section 7.

According to linear stability theory [13] the perturbational fields $\boldsymbol{B} - \boldsymbol{B}_0$ and $p - p_0$ are given by

$$\begin{aligned} \boldsymbol{B} - \boldsymbol{B}_0 &= \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \boldsymbol{B}_0) = \varepsilon \boldsymbol{\nabla} \times (\boldsymbol{\xi}_1 \times \boldsymbol{B}_0) \\ &\quad + \varepsilon^2 \boldsymbol{\nabla} \times (\boldsymbol{\xi}_2 \times \boldsymbol{B}_0) + \cdots \\ + p - p_0 &= -(\boldsymbol{\xi} \cdot \boldsymbol{\nabla} p_0 + \tfrac{5}{3} p_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi}), \\ &= -\varepsilon (\boldsymbol{\xi}_1 \cdot \boldsymbol{\nabla} p_0 + \tfrac{5}{3} p_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi}_1) \\ &\quad - \varepsilon^2 (\boldsymbol{\xi}_2 \cdot \boldsymbol{\nabla} p_0 + \tfrac{5}{3} p_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi}_2) - \cdots, \end{aligned} \tag{17}$$

i.e. $\boldsymbol{B} - \boldsymbol{B}_0$ and $p - p_0$ are power series in $\varepsilon$. The fluid velocity is given by

$$\boldsymbol{v} = \partial \boldsymbol{\xi}/\partial t = \sqrt{\tau}\, \partial \boldsymbol{\xi}/\partial T. \tag{18}$$

Following Eq. (18) we shall represent also nonlinearly the velocity field by a timederivative

$$\boldsymbol{v} = \sqrt{\tau}\, \partial \boldsymbol{\varphi}/\partial T, \tag{19}$$

where the vector field $\boldsymbol{\varphi}$ depends on $\boldsymbol{r}$ and $t$. Let us note that in the nonlinear case $\boldsymbol{\varphi}$ cannot be interpreted as the fluid displacement. The nonlinear relationship between the latter and $\boldsymbol{\varphi}$ will be derived in the next Section.

In order to solve the nonlinear equations of motion we shall try the same scaling as the one obtained in the linear case, and by analogy we shall also expand $\varrho$ in powers of $\varepsilon$:

$$\begin{aligned}
\varrho - \varrho_{00} &= \varepsilon \varrho_1 + \cdots, \\
\boldsymbol{B} - \boldsymbol{B}_0 &= \varepsilon \boldsymbol{B}_1 + \varepsilon^2 \boldsymbol{B}_2 + \varepsilon^3 \boldsymbol{B}_3 + \cdots, \\
p - p_0 &= \varepsilon p_1 + \varepsilon^2 p_2 + \varepsilon^3 p_3 + \cdots, \\
\boldsymbol{\varphi} &= \varepsilon \boldsymbol{\varphi}_1 + \varepsilon^2 \boldsymbol{\varphi}_2 + \varepsilon^3 \boldsymbol{\varphi}_3 + \cdots,
\end{aligned}$$

where $\varrho_i$, $\boldsymbol{B}_i$, $p_i$ and $\boldsymbol{\varphi}_i$, $i = 1, 2, 3, \ldots$, are functions of $\boldsymbol{r}$ and $t$. If we introduce here the equilibrium expansions (2) together with expansion (10), we finally obtain the following scaling

$$\begin{aligned}
\varrho &= \varrho_{00} + \varepsilon \varrho_1 + \cdots, \\
\boldsymbol{B} &= \boldsymbol{B}_{00} + \varepsilon(\tau_1 \boldsymbol{B}_{01} + \boldsymbol{B}_1) \\
&\quad + \varepsilon^2(\tau_2 \boldsymbol{B}_{01} + \tau_1{}^2 \boldsymbol{B}_{02} + \boldsymbol{B}_2) \\
&\quad + \varepsilon^3(\tau_3 \boldsymbol{B}_{01} + 2\,\tau_1 \tau_2 \boldsymbol{B}_{02} \\
&\qquad + \tau_1{}^3 \boldsymbol{B}_{03} + \boldsymbol{B}_3) + \cdots, \\
p &= p_{00} + \varepsilon(\tau_1 p_{01} + p_1) \\
&\quad + \varepsilon^2(\tau_2 p_{01} + \tau_1{}^2 p_{02} + p_2) \\
&\quad + \varepsilon^3(\tau_3 p_{01} + 2\,\tau_1 \tau_2 p_{02} \\
&\qquad + \tau_1{}^3 p_{03} + p_3) + \cdots, \\
\boldsymbol{\varphi} &= \varepsilon \boldsymbol{\varphi}_1 + \varepsilon^2 \boldsymbol{\varphi}_2 + \varepsilon^3 \boldsymbol{\varphi}_3 + \cdots, \\
\tau &= \varepsilon \tau_1 + \varepsilon^2 \tau_2 + \varepsilon^3 \tau_3 + \cdots.
\end{aligned} \tag{20}$$

The velocity field $\boldsymbol{v}$ is given by (19), and time derivatives scale according to (16).

## 4. Resolution of the Nonlinear Relation Between $\boldsymbol{\varphi}$ and $\boldsymbol{\xi}$

Let $\boldsymbol{R}(t)$ be the position of a fluid element at time $t$ the equilibrium position of which was $\boldsymbol{r}$. Then,

$$\boldsymbol{\xi}(\boldsymbol{r}, t) = \boldsymbol{R}(t) - \boldsymbol{r}$$

is the fluid displacement, and from $\mathrm{d}\boldsymbol{R}/\mathrm{d}t = \boldsymbol{v}$ we get the nonlinear relationship

$$\partial \boldsymbol{\xi}(\boldsymbol{r}, t)/\partial t = \boldsymbol{v}(\boldsymbol{r} + \boldsymbol{\xi}(\boldsymbol{r}, t), t)\,. \tag{21}$$

With the ansatz (19) and using relation (16), we obtain after one time integration

$$\boldsymbol{\xi}(\boldsymbol{r}, t) = \boldsymbol{\varphi}(\boldsymbol{r} + \boldsymbol{\xi}(\boldsymbol{r}, t), t)\,. \tag{22}$$

An integration function of $\boldsymbol{r}$ has been chosen in such a way that $\boldsymbol{\xi} = 0$ for $\boldsymbol{\varphi} = 0$. This is possible since in the definition of $\boldsymbol{\varphi}$ (E. (19)) a free vector function of $\boldsymbol{r}$ remains undetermined.

Expansion of $\boldsymbol{\varphi}(\boldsymbol{r} + \boldsymbol{\xi}, t)$ in a Taylor series with respect to $\boldsymbol{\xi}$ yields

$$\boldsymbol{\xi} = \boldsymbol{\varphi} + \boldsymbol{\xi} \cdot \nabla \boldsymbol{\varphi} + \frac{1}{2} \boldsymbol{\xi}\boldsymbol{\xi} : \frac{\partial}{\partial \boldsymbol{r}} \frac{\partial}{\partial \boldsymbol{r}} \boldsymbol{\varphi} + \cdots, \tag{23}$$

where the scalar operator $\boldsymbol{\xi}\boldsymbol{\xi}$: $\partial/\partial\boldsymbol{r}\, \partial/\partial\boldsymbol{r}$ is the scalar product of two tensors. In (23), the argument of the functions on both sides are $\boldsymbol{r}$ and $t$. This nonlinear relationship between $\boldsymbol{\varphi}$ and $\boldsymbol{\xi}$ may be solved by using the expansion (20) for $\boldsymbol{\varphi}$ and by assuming a similar expansion for $\boldsymbol{\xi}$

$$\boldsymbol{\xi} = \varepsilon \boldsymbol{\xi}_1 + \varepsilon^2 \boldsymbol{\xi}_2 + \varepsilon^3 \boldsymbol{\xi}_3 + \cdots. \tag{24}$$

Equating equal powers of $\varepsilon$ on both sides, we obtain the equations

$$\begin{aligned}
&\boldsymbol{\xi}_1 = \boldsymbol{\varphi}_1\,, \quad \boldsymbol{\xi}_2 = \boldsymbol{\varphi}_2 + \boldsymbol{\xi}_1 \cdot \nabla \boldsymbol{\varphi}_1\,, \\
&\boldsymbol{\xi}_3 = \boldsymbol{\varphi}_3 + \boldsymbol{\xi}_1 \cdot \nabla \boldsymbol{\varphi}_2 + \boldsymbol{\xi}_2 \cdot \nabla \boldsymbol{\varphi}_1 \\
&\qquad + \frac{1}{2} \boldsymbol{\xi}_1 \boldsymbol{\xi}_1 : \frac{\partial}{\partial \boldsymbol{r}} \frac{\partial}{\partial \boldsymbol{r}} \boldsymbol{\varphi}_1\,.
\end{aligned}$$

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If the lower order results are introduced into the higher order results, we finally obtain

$$\begin{aligned}
&\boldsymbol{\xi}_1 = \boldsymbol{\varphi}_1\,, \quad \boldsymbol{\xi}_2 = \boldsymbol{\varphi}_2 + \boldsymbol{\varphi}_1 \cdot \nabla \boldsymbol{\varphi}_1\,, \\
&\boldsymbol{\xi}_3 = \boldsymbol{\varphi}_3 + \boldsymbol{\varphi}_1 \cdot \nabla \boldsymbol{\varphi}_2 + (\boldsymbol{\varphi}_2 + \boldsymbol{\varphi}_1 \cdot \nabla \boldsymbol{\varphi}_1) \cdot \nabla \boldsymbol{\varphi}_1 \\
&\qquad + \frac{1}{2} \boldsymbol{\varphi}_1 \boldsymbol{\varphi}_1 : \frac{\partial}{\partial \boldsymbol{r}} \frac{\partial}{\partial \boldsymbol{r}} \boldsymbol{\varphi}_1\,,
\end{aligned} \tag{25}$$

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As we shall see in Sects. 5—6 the vector functions $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \ldots$ are determined by the nonlinear equations of motion which do not contain the displacement vector $\boldsymbol{\xi}$. Thus, once solutions $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \ldots$ have been obtained using the methods of the next Sections, the displacement field $\boldsymbol{\xi}(\boldsymbol{r}, t)$ can be easily calculated by introducing Eqs. (25) into the expansion (24).

## 5. Reduction of the Nonlinear Equations of Motion

In ideal MHD theory, the nonlinear equations of motion are

$$\begin{aligned}
&\text{a)} && \varrho(\partial \boldsymbol{v}/\partial t + \boldsymbol{v}\cdot\nabla\boldsymbol{v}) = \boldsymbol{j}\times\boldsymbol{B} - \nabla p\,,\\
&\text{b)} && \partial\boldsymbol{B}/\partial t = \nabla\times(\boldsymbol{v}\times\boldsymbol{B})\,,\\
&\text{c)} && \partial p/\partial t = -(\boldsymbol{v}\cdot\nabla p + \tfrac{5}{3}\,p\nabla\cdot\boldsymbol{v})\,,\\
&\text{d)} && \partial\varrho/\partial t = -\nabla\cdot(\varrho\boldsymbol{v})\,.
\end{aligned} \tag{26}$$

$\boldsymbol{j}$ is given by Eq. (7), and $\nabla\cdot\boldsymbol{B} = 0$ must be imposed as initial condition. Since we shall only consider motions which are dynamically connected with equilibrium states satisfying $\nabla\cdot\boldsymbol{B}_0 = 0$, we may forget about this initial condition.

For internal modes, we have the boundary condition

$$\boldsymbol{n}_{00}\cdot\boldsymbol{v} = 0\,, \tag{27}$$

where $\boldsymbol{n}_{00}(\boldsymbol{r})$ is a unit vector normal to the unperturbed plasma boundary. Using the notation $\boldsymbol{n}_{00}$ we tacidly assume that the shape of the plasma boundary does not depend on the driving parameter. The latter possibility will be considered in the context of external modes in a following paper [14].

With our ansatz (19), from Eq. (27) we obtain after time-integration

$$\boldsymbol{n}_{00}\cdot\boldsymbol{\varphi} = f(\boldsymbol{r})\,,$$

where $f(\boldsymbol{r})$ is an arbitrary function. Since in the definition of $\boldsymbol{\varphi}$ (Eq. (19)) only timedependent vector fields $\boldsymbol{\varphi}$ are meaningful, the last equation can only be satisfied if $f(\boldsymbol{r}) \equiv 0$, i.e. we have

$$\boldsymbol{n}_{00}\cdot\boldsymbol{\varphi} = 0\,. \tag{28}$$

Inserting the expansion (20) for $\boldsymbol{\varphi}$ in (28), we obtain the reduced boundary conditions

$$\boldsymbol{n}_{00}\cdot\boldsymbol{\varphi}_i = 0\,,\quad i = 1,2,3,\ldots\,. \tag{29}$$

Let us now insert (16), (19) and (20) into the nonlinear Equations (26). Equating the coefficients of equal powers in $\varepsilon$ and making use of the equilibrium Eqs. (8), with the notations of the appendix we obtain after some calculation the following systems of equations:

*order $\varepsilon$:*

$$\begin{aligned}
&\text{a)} && 0 = \boldsymbol{j}_{00}\times\boldsymbol{B}_1 + \boldsymbol{j}_1\times\boldsymbol{B}_{00} - \nabla p_1\,,\\
&\text{b)*} && \partial\boldsymbol{B}_1/\partial T = \partial\mathbf{B}_{10}(\boldsymbol{\varphi}_1)/\partial T\,,\\
&\text{c)} && \partial p_1/\partial T = \partial\not{p}_{10}(\boldsymbol{\varphi}_1)/\partial T\,,\\
&\text{d)} && \partial\varrho_1/\partial T = -\,\partial(\varrho_{00}\nabla\cdot\boldsymbol{\varphi}_1)/\partial T\,.
\end{aligned} \tag{30}$$

Only the lowest order of the continuity equation will be needed so that its higher orders are going to be omitted.

*Order $\varepsilon^2$:*

$$\begin{aligned}
&\text{a)} && \tau_1\varrho_{00}\,\partial^2\boldsymbol{\varphi}_1/\partial T^2 = \boldsymbol{j}_{00}\times\boldsymbol{B}_2 + \boldsymbol{j}_2\times\boldsymbol{B}_{00} - \nabla p_2 + \tau_1(\boldsymbol{j}_{01}\times\boldsymbol{B}_1 + \boldsymbol{j}_1\times\boldsymbol{B}_{01})\,,\\
&\text{b)} && \partial\boldsymbol{B}_2/\partial T = \partial(\mathbf{B}_{10}(\boldsymbol{\varphi}_2) + \tau_1\mathbf{B}_{11}(\boldsymbol{\varphi}_1))/\partial T + \mathbf{B}_1(\partial\boldsymbol{\varphi}_1/\partial T, \boldsymbol{B}_1)\,,\\
&\text{c)} && \partial p_2/\partial T = \partial(\not{p}_{10}(\boldsymbol{\varphi}_2) + \tau_1\not{p}_{11}(\boldsymbol{\varphi}_1))/\partial T + \not{p}_1(\partial\boldsymbol{\varphi}_1/\partial T, p_1)\,.
\end{aligned} \tag{31}$$

*Order $\varepsilon^3$:*

$$\begin{aligned}
&\text{a)} && \tau_2\varrho_{00}\,\partial^2\boldsymbol{\varphi}_1/\partial T^2 + \tau_1\varrho_{00}[\partial^2\boldsymbol{\varphi}_2/\partial T^2 + (\partial\boldsymbol{\varphi}_1/\partial T)\cdot\nabla(\partial\boldsymbol{\varphi}_1/\partial T)] + \tau_1\varrho_1\,\partial^2\boldsymbol{\varphi}_1/\partial T^2\\
& && = \boldsymbol{j}_{00}\times\boldsymbol{B}_3 + \boldsymbol{j}_3\times\boldsymbol{B}_{00} - \nabla p_3 + \tau_1(\boldsymbol{j}_{01}\times\boldsymbol{B}_2 + \boldsymbol{j}_2\times\boldsymbol{B}_{01})\\
& && \quad + \tau_2(\boldsymbol{j}_{01}\times\boldsymbol{B}_1 + \boldsymbol{j}_1\times\boldsymbol{B}_{01}) + \tau_1^2(\boldsymbol{j}_{02}\times\boldsymbol{B}_1 + \boldsymbol{j}_1\times\boldsymbol{B}_{02}) + \boldsymbol{j}_1\times\boldsymbol{B}_2 + \boldsymbol{j}_2\times\boldsymbol{B}_1\,,\\
&\text{b)} && \partial\boldsymbol{B}_3/\partial T = (\partial/\partial T)[\mathbf{B}_{10}(\boldsymbol{\varphi}_3) + \tau_1\mathbf{B}_{11}(\boldsymbol{\varphi}_2) + \tau_2\mathbf{B}_{11}(\boldsymbol{\varphi}_1) + \tau_1^2\mathbf{B}_{12}(\boldsymbol{\varphi}_1)] + \mathbf{B}_1(\partial\boldsymbol{\varphi}_1/\partial T, \boldsymbol{B}_2)\\
& && \quad + \mathbf{B}_1(\partial\boldsymbol{\varphi}_2/\partial T, \boldsymbol{B}_1)\,,\\
&\text{c)} && \partial p_3/\partial T = (\partial/\partial T)[\not{p}_{10}(\boldsymbol{\varphi}_3) + \tau_1\not{p}_{11}(\boldsymbol{\varphi}_2) + \tau_2\not{p}_{11}(\boldsymbol{\varphi}_1) + \tau_1^2\not{p}_{12}(\boldsymbol{\varphi}_1)] + \not{p}_1(\partial\boldsymbol{\varphi}_1/\partial T, p_2)\\
& && \quad + \not{p}_1(\partial\boldsymbol{\varphi}_2/\partial T, p_1)\,.
\end{aligned} \tag{32}$$

In order to obtain information about the nonlinear motion, it will turn out that, depending on the symmetry of the problem, we shall have to go either to second or to third order in $\varepsilon$. Therefore, we shall not consider higher orders in this paper.

* Notation see Appendix a).

## 9. Derivation of Nonlinear amplitude Equations for $\boldsymbol{\varphi}_1$

The next step will be to integrate the equations for $\boldsymbol{B}_i$ and $p_i$ with respect to time and to eliminate these quantities order for order. The condition for dynamical connection of the timedependent states with equilibrium is

$$B_i{}^2 = p_i = 0\,, \quad i = 1, 2, 3, \ldots \tag{33}$$

for $\boldsymbol{\xi} = 0$. In the following, integration functions will be chosen such that the conditions (33) be satisfied for $\boldsymbol{\varphi}_1 = \boldsymbol{\varphi}_2 = \boldsymbol{\varphi}_3 = 0$. According to (24), (25) we then have $\boldsymbol{\xi} = 0$ too.

*Order $\varepsilon$:*

Time integration of Eqs. (30) b)—c) yields

$$\boldsymbol{B}_1 = \mathbf{B}_{10}(\boldsymbol{\varphi}_1)\,, \quad p_1 = \not{p}_{10}(\boldsymbol{\varphi}_1)\,. \tag{34}$$

Inserting this result into (30a), using (A8) we obtain

$$\mathbf{F}_{00}(\boldsymbol{\varphi}_1) = 0\,. \tag{35}$$

(30d) may be integrated giving

$$\varrho_1 = -\varrho_{00}\,\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}_1\,. \tag{36}$$

Since the MHD-operator $\mathbf{F}_{00}$ contains only spatial derivatives, we may separate the space- and time-dependencies in $\boldsymbol{\varphi}_1$ by setting

$$\boldsymbol{\varphi}_1(\boldsymbol{r}, T) = A_1(T)\,\boldsymbol{\Phi}_1(\boldsymbol{r})^{**}. \tag{37}$$

Inserting this in (34), with the property (A14) we get

$$\begin{aligned} \boldsymbol{B}_1 &= A_1(T)\,\tilde{\boldsymbol{B}}_1(\boldsymbol{r})\,, \quad \tilde{\boldsymbol{B}}_1 = \mathbf{B}_{10}(\boldsymbol{\Phi}_1)\,, \\ p_1 &= A_1(T)\,\tilde{p}_1(\boldsymbol{r})\,, \quad \tilde{p}_1 = \not{p}_{10}(\boldsymbol{\Phi}_1)\,. \end{aligned} \tag{38}$$

Our separation ansatz (37) enables a time integration of the higher order equations.

*Order $\varepsilon^2$:*

With (37), (38), and (A16), the last terms in Eqs. (31) b)—c) become also time-derivatives:

$$\begin{aligned} \mathbf{B}_1(\partial\boldsymbol{\varphi}_1/\partial T, \boldsymbol{B}_1) &= A_1\dot{A}_1\,\mathbf{B}_1(\boldsymbol{\Phi}_1, \tilde{\boldsymbol{B}}_1) \\ &= (A_1{}^2/2)\cdot\mathbf{B}_1(\boldsymbol{\Phi}_1, \tilde{\boldsymbol{B}}_1) \\ &= (1/2)\,\partial\mathbf{B}_1(\boldsymbol{\varphi}_1, \boldsymbol{B}_1)/\partial T\,. \end{aligned}$$

Similarly we may treat the last term in (31c) and, taking (34) into account, we finally obtain

$$\mathbf{B}_1(\partial\boldsymbol{\varphi}_1/\partial T, \boldsymbol{B}_1) = (1/2)\,\partial\mathbf{B}_1(\boldsymbol{\varphi}_1, \mathbf{B}_{10}(\boldsymbol{\varphi}_1))/\partial T\,,$$

$$\not{p}_1(\partial\boldsymbol{\varphi}_1/\partial T, p_1) = (1/2)\,\partial\not{p}_1(\boldsymbol{\varphi}_1, \not{p}_{10}(\boldsymbol{\varphi}_1))/\partial T\,,$$

** In view of Eqs. (38), using $\tilde{\boldsymbol{\varphi}}_1$ instead of $\boldsymbol{\Phi}_1$ would yield a more uniform notation. The choice of $\boldsymbol{\Phi}_1$ is a printing technicality.

and together with (31) b)—c)

$$\begin{aligned} \boldsymbol{B}_2 &= \mathbf{B}_{10}(\boldsymbol{\varphi}_2) + \tau_1\,\mathbf{B}_{11}(\boldsymbol{\varphi}_1) \\ &\quad + (1/2)\,\mathbf{B}_1(\boldsymbol{\varphi}_1, \mathbf{B}_{10}(\boldsymbol{\varphi}_1))\,, \end{aligned} \tag{39}$$

$$\begin{aligned} p_2 &= \not{p}_{10}(\boldsymbol{\varphi}_2) + \tau_1\,\not{p}_{11}(\boldsymbol{\varphi}_1) \\ &\quad + (1/2)\,\not{p}_1(\boldsymbol{\varphi}_1, \not{p}_{10}(\boldsymbol{\varphi}_1))\,. \end{aligned}$$

Inserting the results (34) and (39) into (31a) we obtain with the definitions (A9) and (A11)

$$\begin{aligned} \mathbf{F}_{00}(\boldsymbol{\varphi}_2) &= \tau_1\,\varrho_{00}\,\partial^2\boldsymbol{\varphi}_1/\partial T^2 - \tau_1\,\mathbf{F}_{01}(\boldsymbol{\varphi}_1) \\ &\quad - (1/2)\,\mathbf{G}_{00}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_1)\,. \end{aligned} \tag{40}$$

(40) is an inhomogeneous differential equation for $\boldsymbol{\varphi}_2$. A solution exists only when its right hand side is orthogonal to the solution of the adjoint homogeneous problem. The latter is given by (35), since $\mathbf{F}_{00}$ is selfadjoint for the boundary conditions (29). Using (15) with $\boldsymbol{\xi}_i$ replaced by $\boldsymbol{\varphi}_i$, Eqs. (37), (A16) and introducing the normalization

$$(\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_1) = 1\,, \tag{41}$$

for (40) we obtain the condition of integrability

$$\tau_1(\ddot{A}_1 - \gamma_{\mathrm{T}}{}^2 A_1) = (3\,\delta_{\mathrm{T}}/2)\,A_1{}^2\,, \tag{42}$$

where

$$\gamma_{\mathrm{T}} = \varrho_{00}^{-1}\,(\boldsymbol{\Phi}_1, \mathbf{F}_{01}(\boldsymbol{\Phi}_1))\,, \tag{43}$$

$$\delta_{\mathrm{T}} = (3\,\varrho_{00})^{-1}\,(\boldsymbol{\Phi}_1, \mathbf{G}_{00}(\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_1))\,. \tag{44}$$

Inserting (37) and (42) into (40), we get using (A16)

$$\begin{aligned} \mathbf{F}_{00}(\boldsymbol{\varphi}_2) &= \tau_1\,(\varrho_{00}\,\gamma_{\mathrm{T}}{}^2\,\boldsymbol{\Phi}_1 - \mathbf{F}_{01}(\boldsymbol{\Phi}_1))\,A_1 \\ &\quad + (3\,\varrho_{00}\,\delta_{\mathrm{T}}\,\boldsymbol{\Phi}_1 - \mathbf{G}_{00}(\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_1))\,(A_1{}^2/2)\,. \end{aligned} \tag{45}$$

From this equation, we obtain the timedependence of $\boldsymbol{\varphi}_2$ using the ansatz

$$\begin{aligned} \boldsymbol{\varphi}_2 &= \boldsymbol{\varphi}_{20} + \boldsymbol{\varphi}_{21} + \boldsymbol{\varphi}_{22}\,, \\ \boldsymbol{\varphi}_{20} &= A_2(T)\,\boldsymbol{\Phi}_{20}(\boldsymbol{r})\,, \\ \boldsymbol{\varphi}_{21} &= \tau_1 A_1(T)\,\boldsymbol{\Phi}_{21}(\boldsymbol{r})\,, \\ \boldsymbol{\varphi}_{22} &= (A_1{}^2(T)/2)\,\boldsymbol{\Phi}_{22}(\boldsymbol{r})\,. \end{aligned} \tag{46}$$

In order to satisfy the boundary condition $\boldsymbol{n}_{00}\cdot\boldsymbol{\varphi}_2 = 0$ (Eq. (29)) we require

$$\boldsymbol{n}_{00}\cdot\boldsymbol{\varphi}_{2n} = 0\,, \quad n = 0, 1, 2\,. \tag{47}$$

This requirement is necessary for

$$A_2(T) \not\equiv A_1(T) \not\equiv (A_1{}^2(T)/2)$$

which will generally be the case.

According to (45) and (A14), the $\mathbf{\Phi}_{2n}$ must satisfy the equations

$$\begin{aligned} &\text{a)} \quad \mathbf{F}_{00}(\mathbf{\Phi}_{20}) = 0, \\ &\text{b)} \quad \mathbf{F}_{00}(\mathbf{\Phi}_{21}) = \varrho_{00}\gamma_T^2 \mathbf{\Phi}_1 - \mathbf{F}_{01}(\mathbf{\Phi}_1), \\ &\text{c)} \quad \mathbf{F}_{00}(\mathbf{\Phi}_{22}) = 3\varrho_{00}\delta_T \mathbf{\Phi}_1 - \mathbf{G}_{00}(\mathbf{\Phi}_1, \mathbf{\Phi}_1). \end{aligned} \tag{48}$$

According to the definitions of $\gamma_T^2$ and $\delta_T$ (Eqs. (43) to (44)), the conditions of integrability for the inhomogeneous equations (48) b)—c) are automatically satisfied. Obviously, for internal modes we have

$$\mathbf{\Phi}_{20}(\boldsymbol{r}) = \mathbf{\Phi}_1(\boldsymbol{r}). \tag{49}$$

We shall however keep the notation $\mathbf{\Phi}_{20}$ for the sake of generality. Inserting (37) and (46) in (39), we obtain using (A14), (A16)

$$\begin{aligned} \boldsymbol{B}_2 &= \boldsymbol{B}_{20} + \boldsymbol{B}_{21} + \boldsymbol{B}_{22}, \\ p_2 &= p_{20} + p_{21} + p_{22}, \\ \boldsymbol{B}_{20} &= A_2(T)\,\tilde{\boldsymbol{B}}_{20}(\boldsymbol{r}), \\ \boldsymbol{B}_{21} &= \tau_1 A_1(T)\,\tilde{\boldsymbol{B}}_{21}(\boldsymbol{r}), \\ \boldsymbol{B}_{22} &= (A_1^2(T)/2)\,\tilde{\boldsymbol{B}}_{22}(\boldsymbol{r}), \\ p_{20} &= A_2(T)\,\tilde{p}_{20}(\boldsymbol{r}), \\ p_{21} &= \tau_1 A_1(T)\,\tilde{p}_{21}(\boldsymbol{r}), \\ p_{22} &= (A_1^2(T)/2)\,\tilde{p}_{22}(\boldsymbol{r}), \end{aligned} \tag{50}$$

where

$$\begin{aligned} \tilde{\boldsymbol{B}}_{20} &= \mathbf{B}_{10}(\mathbf{\Phi}_{20}), \quad p_{20} = \not{p}_{10}(\mathbf{\Phi}_{20}), \\ \tilde{\boldsymbol{B}}_{21} &= \mathbf{B}_{10}(\mathbf{\Phi}_{21}) + \mathbf{B}_{11}(\mathbf{\Phi}_1), \\ p_{21} &= \not{p}_{10}(\mathbf{\Phi}_{21}) + \not{p}_{11}(\mathbf{\Phi}_1), \\ \tilde{\boldsymbol{B}}_{22} &= \mathbf{B}_{10}(\mathbf{\Phi}_{22}) + \mathbf{B}_1(\mathbf{\Phi}_1, \mathbf{B}_{10}(\mathbf{\Phi}_1)), \\ p_{22} &= \not{p}_{10}(\mathbf{\Phi}_{22}) + \not{p}_1(\mathbf{\Phi}_1, \not{p}_{10}(\mathbf{\Phi}_1)). \end{aligned} \tag{51}$$

*Order $\varepsilon^3$:*

Let us now treat the time integration of (32b). Using (37), (38), (50), and (A16), we obtain

$$\begin{aligned} &\mathbf{B}_1(\partial\boldsymbol{\varphi}_1/\partial T, \boldsymbol{B}_2) + \mathbf{B}_1(\partial\boldsymbol{\varphi}_2/\partial T, \boldsymbol{B}_1) \\ &= \dot{A}_1 A_2 \mathbf{B}_1(\mathbf{\Phi}_1, \tilde{\boldsymbol{B}}_{20}) + \dot{A}_2 A_1 \mathbf{B}_1(\mathbf{\Phi}_{20}, \tilde{\boldsymbol{B}}_1) \\ &\quad + \tau_1 (A_1^2/2)^{\cdot} [\mathbf{B}_1(\mathbf{\Phi}_1, \tilde{\boldsymbol{B}}_{21}) \\ &\quad + \mathbf{B}_1(\mathbf{\Phi}_{21}, \tilde{\boldsymbol{B}}_1)] \\ &\quad + (A_1^3/6)^{\cdot} [\mathbf{B}_1(\mathbf{\Phi}_1, \tilde{\boldsymbol{B}}_{22}) + 2\,\mathbf{B}_1(\mathbf{\Phi}_{22}, \tilde{\boldsymbol{B}}_1)]. \end{aligned}$$

Defining functions $a_{12}(T)$ and $a_{21}(T)$ through

$$\dot{a}_{12} = \dot{A}_1 A_2, \quad \dot{a}_{21} = \dot{A}_2 A_1 \tag{52}$$

and using (51) and (A16), we obtain after some calculation

$$\begin{aligned} &\mathbf{B}_1(\partial\boldsymbol{\varphi}_1/\partial T, \boldsymbol{B}_2) + \mathbf{B}_1(\partial\boldsymbol{\varphi}_2/\partial T, \boldsymbol{B}_1) \\ &= (\partial/\partial T)\{a_{12}\mathbf{B}_1(\mathbf{\Phi}_1, \mathbf{B}_{10}(\mathbf{\Phi}_{20})) + a_{21}\mathbf{B}_1(\mathbf{\Phi}_{20}, \mathbf{B}_{10}(\mathbf{\Phi}_1)) \\ &\quad + \tfrac{1}{2}[\mathbf{B}_1(\boldsymbol{\varphi}_1, \mathbf{B}_{10}(\boldsymbol{\varphi}_{21})) + \mathbf{B}_1(\boldsymbol{\varphi}_{21}, \mathbf{B}_{10}(\boldsymbol{\varphi}_1)) + \tau_1 \mathbf{B}_1(\boldsymbol{\varphi}_1, \mathbf{B}_{11}(\boldsymbol{\varphi}_1))] \\ &\quad + \tfrac{1}{3}[\mathbf{B}_1(\boldsymbol{\varphi}_1, \mathbf{B}_{10}(\boldsymbol{\varphi}_{22})) + 2\,\mathbf{B}_1(\boldsymbol{\varphi}_{22}, \boldsymbol{B}_{10}(\boldsymbol{\varphi}_1)) + \tfrac{1}{2}\mathbf{B}_1(\boldsymbol{\varphi}_1, \mathbf{B}_1(\boldsymbol{\varphi}_1, \mathbf{B}_{10}(\boldsymbol{\varphi}_1)))]\}. \end{aligned}$$

We can now perform the timeintegration of (32) b) getting

$$\begin{aligned} \boldsymbol{B}_3 &= \mathbf{B}_{10}(\boldsymbol{\varphi}_3) + \tau_1 \mathbf{B}_{11}(\boldsymbol{\varphi}_2) + \tau_2 \mathbf{B}_{11}(\boldsymbol{\varphi}_1) + \tau_1^2 \mathbf{B}_{12}(\boldsymbol{\varphi}_1) + a_{12}\mathbf{B}_1(\mathbf{\Phi}_1, \mathbf{B}_{10}(\mathbf{\Phi}_{20})) \\ &\quad + a_{21}\mathbf{B}_1(\mathbf{\Phi}_{20}, \mathbf{B}_{10}(\mathbf{\Phi}_1)) + \tfrac{1}{2}[\mathbf{B}_1(\boldsymbol{\varphi}_1, \mathbf{B}_{10}(\boldsymbol{\varphi}_{21})) + \mathbf{B}_1(\boldsymbol{\varphi}_{21}, \mathbf{B}_{10}(\boldsymbol{\varphi}_1)) + \tau_1 \mathbf{B}_1(\boldsymbol{\varphi}_1, \mathbf{B}_{11}(\boldsymbol{\varphi}_1))] \\ &\quad + \tfrac{1}{3}[\mathbf{B}_1(\boldsymbol{\varphi}_1, \mathbf{B}_{10}(\boldsymbol{\varphi}_{22})) + 2\,\mathbf{B}_1(\boldsymbol{\varphi}_{22}, \mathbf{B}_{10}(\boldsymbol{\varphi}_1)) + \tfrac{1}{2}\mathbf{B}_1(\boldsymbol{\varphi}_1, \mathbf{B}_1(\boldsymbol{\varphi}_1, \mathbf{B}_{10}(\boldsymbol{\varphi}_1)))]. \end{aligned} \tag{53}$$

We obtain in a similar way

$$\begin{aligned} p_3 &= \not{p}_{10}(\boldsymbol{\varphi}_3) + \tau_1 \not{p}_{11}(\boldsymbol{\varphi}_2) + \tau_2 \not{p}_{11}(\boldsymbol{\varphi}_1) + \tau_1^2 \not{p}_{12}(\boldsymbol{\varphi}_1) + a_{12}\not{p}_1(\mathbf{\Phi}_1, \not{p}_{10}(\mathbf{\Phi}_{20})) \\ &\quad + a_{21}\not{p}_1(\mathbf{\Phi}_{20}, \not{p}_{10}(\mathbf{\Phi}_1)) + \tfrac{1}{2}[\not{p}_1(\boldsymbol{\varphi}_1, \not{p}_{10}(\boldsymbol{\varphi}_{21})) + \not{p}_1(\boldsymbol{\varphi}_{21}, \not{p}_{10}(\boldsymbol{\varphi}_1)) + \tau_1 \not{p}_1(\boldsymbol{\varphi}_1, \not{p}_{11}(\boldsymbol{\varphi}_1))] \\ &\quad + \tfrac{1}{3}[\not{p}_1(\boldsymbol{\varphi}_1, \not{p}_{10}(\boldsymbol{\varphi}_{22})) + 2\not{p}_1(\boldsymbol{\varphi}_{22}, \not{p}_{10}(\boldsymbol{\varphi}_1)) + \tfrac{1}{2}\not{p}_1(\boldsymbol{\varphi}_1, \not{p}_1(\boldsymbol{\varphi}_1, \not{p}_{10}(\boldsymbol{\varphi}_1)))]. \end{aligned} \tag{54}$$

From Eqs. (52), we get $\mathrm{d}(a_{12} + a_{21} - A_1 A_2)/\mathrm{d}T = 0$ or

$$a_{12} + a_{21} = A_1 A_2, \tag{55}$$

where the integration constant is chosen such that $B_3^2 = p_3 = 0$ for $\boldsymbol{\varphi}_3 = 0$ and $A_1 = A_2 = 0$.

With the help of (36), (53), and (54), $\varrho_1$, $\boldsymbol{B}_3$ and $p_3$ may be eliminated from (32a). After a rather lengthy calculation where use is made of (39, (55) and of several operators defined in the Appendix together with their properties described there, one obtains

$$\begin{aligned}\mathbf{F}_{00}(\boldsymbol{\varphi}_3) = \varrho_{00}\{&\tau_2\,\partial^2\boldsymbol{\varphi}_1/\partial T^2 + \tau_1[\partial^2\boldsymbol{\varphi}_2/\partial T^2 + (\partial\boldsymbol{\varphi}_1/\partial T)\cdot\boldsymbol{\nabla}(\partial\boldsymbol{\varphi}_1/\partial T) - \partial^2\boldsymbol{\varphi}_1/\partial T^2\,\nabla\cdot\boldsymbol{\varphi}_1]\}\\ &- \{\tau_1\mathbf{F}_{01}(\boldsymbol{\varphi}_2) + \tau_2\mathbf{F}_{01}(\boldsymbol{\varphi}_1) + \tau_1{}^2\mathbf{F}_{02}(\boldsymbol{\varphi}_1) + a_{12}\,\mathbf{G}_{00}(\boldsymbol{\Phi}_1,\boldsymbol{\Phi}_{20}) + a_{21}\,\mathbf{G}_{00}(\boldsymbol{\Phi}_{20},\boldsymbol{\Phi}_1)\\ &+ \tfrac{1}{2}[\mathbf{G}_{00}(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_{21}) + \mathbf{G}_{00}(\boldsymbol{\varphi}_{21},\boldsymbol{\varphi}_1)] + \tfrac{1}{3}[\mathbf{G}_{00}(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_{22}) + 2\,\mathbf{G}_{00}(\boldsymbol{\varphi}_{22},\boldsymbol{\varphi}_1)]\\ &+ (\tau_1/2)\,\mathbf{G}_{01}(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_1) + \tfrac{1}{6}\mathbf{H}_{00}(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_1)\}\,. \qquad (56)\end{aligned}$$

Inserting (37), (46) and using some of the operator properties specified in the Appendix, from this we obtain

$$\begin{aligned}\mathbf{F}_{00}(\boldsymbol{\varphi}_3) = \varrho_{00}\{&\tau_2\ddot{A}_1\,\boldsymbol{\Phi}_1 + \tau_1[\ddot{A}_2\,\boldsymbol{\Phi}_{20} + \tau_1\ddot{A}_1\,\boldsymbol{\Phi}_{21} + \dot{A}_1{}^2(\boldsymbol{\Phi}_{22} + \boldsymbol{\Phi}_1\cdot\boldsymbol{\nabla}\boldsymbol{\Phi}_1)\\ &+ A_1\ddot{A}_1(\boldsymbol{\Phi}_{22} - \boldsymbol{\Phi}_1\,\nabla\cdot\boldsymbol{\Phi}_1)]\}\\ &- \{\tau_1\mathbf{F}_{01}(\boldsymbol{\Phi}_{20})A_2 + \mathbf{G}_{00}(\boldsymbol{\Phi}_1,\boldsymbol{\Phi}_{20})a_{12} + \mathbf{G}_{00}(\boldsymbol{\Phi}_{20},\boldsymbol{\Phi}_1)a_{21}\\ &+ [\tau_1{}^2(\mathbf{F}_{01}(\boldsymbol{\Phi}_{21}) + \mathbf{F}_{02}(\boldsymbol{\Phi}_1)) + \tau_2\mathbf{F}_{01}(\boldsymbol{\Phi}_1)]A_1\\ &+ \tau_1[\mathbf{F}_{01}(\boldsymbol{\Phi}_{22}) + \mathbf{G}_{01}(\boldsymbol{\Phi}_1,\boldsymbol{\Phi}_1) + \mathbf{G}_{00}(\boldsymbol{\Phi}_1,\boldsymbol{\Phi}_{21}) + \mathbf{G}_{00}(\boldsymbol{\Phi}_{21},\boldsymbol{\Phi}_1)]A_1{}^2/2\\ &+ [\mathbf{G}_{00}(\boldsymbol{\Phi}_1,\boldsymbol{\Phi}_{22}) + 2\,\mathbf{G}_{00}(\boldsymbol{\Phi}_{22},\boldsymbol{\Phi}_1) + \mathbf{H}_{00}(\boldsymbol{\Phi}_1,\boldsymbol{\Phi}_1,\boldsymbol{\Phi}_1)]A_1{}^3/6\}\,. \qquad (57)\end{aligned}$$

(57) is generally valid for internal as well as for external modes. A condition of integrability can be derived in a similar way as for (40). Turning ourselves to the special case of internal modes we shall now take advantage of (49). Making use also of (41), (43), and (44), we get the integrability condition

$$\begin{aligned}&\tau_2(\ddot{A}_1 - \gamma_T{}^2A_1) + \tau_1(\ddot{A}_2 - \gamma_T{}^2A_2)\\ &\quad+\tau_1(\tau_1a_1\ddot{A}_1 + a_2\ddot{A}_1A_1 + b\dot{A}_1{}^2)\\ &= 3\,\delta_T A_1A_2 + \tau_1(\tau_1c_1A_1 + c_2A_1{}^2)\\ &\quad+ 2\,\vartheta_T A_1{}^3\,, \qquad (58)\end{aligned}$$

$$\begin{aligned}a_1 &= (\boldsymbol{\Phi}_1,\boldsymbol{\Phi}_{21})\,,\\ a_2 &= (\boldsymbol{\Phi}_1, -\boldsymbol{\Phi}_1\cdot\boldsymbol{\nabla}\boldsymbol{\Phi}_1 + \boldsymbol{\Phi}_{22})\,,\\ b &= (\boldsymbol{\Phi}_1,\boldsymbol{\Phi}_1\cdot\boldsymbol{\nabla}\boldsymbol{\Phi}_1 + \boldsymbol{\Phi}_{22})\,, \qquad (59)\\ c_1 &= \varrho_{00}^{-1}(\boldsymbol{\Phi}_1,\mathbf{F}_{01}(\boldsymbol{\Phi}_{21}) + \mathbf{F}_{02}(\boldsymbol{\Phi}_1))\,,\\ c_2 &= (2\,\varrho_{00})^{-1}(\boldsymbol{\Phi}_1,\mathbf{F}_{01}(\boldsymbol{\Phi}_{22}) + \mathbf{G}_{01}(\boldsymbol{\Phi}_1,\boldsymbol{\Phi}_1)\\ &\quad+ \mathbf{G}_{00}(\boldsymbol{\Phi}_1,\boldsymbol{\Phi}_{21}) + \mathbf{G}_{00}(\boldsymbol{\Phi}_{21},\boldsymbol{\Phi}_1))\,,\\ \vartheta_T &= (12\,\varrho_{00})^{-1}(\boldsymbol{\Phi}_1,\mathbf{G}_{00}(\boldsymbol{\Phi}_1,\boldsymbol{\Phi}_{22})\\ &\quad+ 2\,\mathbf{G}_{00}(\boldsymbol{\Phi}_{22},\boldsymbol{\Phi}_1) + \mathbf{H}_{00}(\boldsymbol{\Phi}_1,\boldsymbol{\Phi}_1,\boldsymbol{\Phi}_1)).\end{aligned}$$

## 7. Nonlinear Theory of Linearly Stable Modes

Due to the presence of $\sqrt{\tau}$ in the time scaling (16) our theory was restricted so far to linearly unstable modes. For linearly stable modes we temporarily redefine $\tau$ replacing Eq. (3) by

$$\tau = \lambda_0 - \lambda \qquad (3\text{a})$$

which produces a change of sign in the linear and third order terms of (2). All expansions with respect to $\varepsilon$ will be kept as in the case $\lambda > \lambda_0$. Technically, this can be achieved by simply changing the sign of $\boldsymbol{B}_{01}$, $\boldsymbol{B}_{03}$, $p_{01}$, $p_{03}$ in the scaling (20) and all subsequent Equations. According to the Appendix we have then to change the signs of $\mathbf{F}_{01}$ and $\mathbf{G}_{01}$. To lowest order, nothing is changed and we get (35) again for the marginal mode. Changes occur in the second and third order Eqs. (40) and (57). Instead of the integrability condition (42) we now get

$$\tau_1(\ddot{A}_1 + \gamma_T{}^2A_1) = (3\,\delta_T/2)A_1{}^2\,, \qquad (42\text{a})$$

where the definitions of $\gamma_T{}^2$ and $\delta_T$ are the same as for $\lambda > \lambda_0$. Instead of (58) we have

$$\begin{aligned}&\tau_2(\ddot{A}_1 + \gamma_T{}^2A_1) + \tau_1(\ddot{A}_2 + \gamma_T{}^2A_2)\\ &\quad+ \tau_1(\tau_1a_1\ddot{A}_1 + a_2A_1\ddot{A}_1 + b\dot{A}_1{}^2)\\ &= 3\,\delta_T A_1A_2 + \tau_1(\tau_1\hat{c}_1A_1 + \hat{c}_2A_1{}^2)\\ &\quad+ 2\,\vartheta_T A_1{}^3\,, \qquad (58\text{a})\end{aligned}$$

where $a_1$, $a_2$, $b$, $\vartheta_T$ are defined as before and where

$$\begin{aligned}\hat{c}_1 &= \varrho_{00}^{-1}(\boldsymbol{\Phi}_1, -\mathbf{F}_{01}(\boldsymbol{\Phi}_{21}) + \mathbf{F}_{02}(\boldsymbol{\Phi}_1))\,,\\ \hat{c}_2 &= (2\,\varrho_{00})^{-1}(\boldsymbol{\Phi}_1, -\mathbf{F}_{01}(\boldsymbol{\Phi}_{22}) - \mathbf{G}_{01}(\boldsymbol{\Phi}_1,\boldsymbol{\Phi}_1)\\ &\quad+ \mathbf{G}_{00}(\boldsymbol{\Phi}_1,\boldsymbol{\Phi}_{21}) + \mathbf{G}_{00}(\boldsymbol{\Phi}_{21},\boldsymbol{\Phi}_1))\,. \qquad (59\text{a})\end{aligned}$$

## 8. Discussion of the Nonlinear Motion

Let us now discuss the physical consequences of the Equations derived. Two mainly different cases must be distinguished.

1. If $\delta_T \neq 0$, (42) is a nonlinear Equation for the first order amplitude of the marginal mode $\boldsymbol{\varphi}_1$. Since according to (43)–(44) $\gamma_T{}^2$ and $\delta_T$ are obtained from $\boldsymbol{\Phi}_1$ by differentiations and an integration, in principle only $\boldsymbol{\Phi}_1$ must be determined from Eq. (35) in order to obtain nonlinear information.

2. If $\delta_T = 0$, Eqs. (42) and (58) reduce to a set of linear Equations of the form

$$\begin{aligned} \tau_1(\ddot{A}_1 - \gamma_T{}^2 A_1) &= 0\,, \\ \tau_1(\ddot{A}_2 - \gamma_T{}^2 A_2) &= f(A_1) \\ &= f(C \exp(\gamma_T T))\,, \end{aligned} \tag{60}$$

and at first glance no nonlinear information can be extracted from the theory. This form turns out to be inappropriate although still valid. The situation is similar to the case of expanding

$$\sin t = t - t^3/3! + \cdots$$

which is useless for long term predictions. However, (42) can be satisfied in this case also by setting

$$\tau_1 = 0\,. \tag{61}$$

Then, (40) reduces to

$$\mathbf{F}_{00}(\boldsymbol{\varphi}_2) = -(1/2)\,\mathbf{G}_{00}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_1)\,. \tag{62}$$

By inspection of (44) we see that the integrability condition for this Equation is just $\delta_T = 0$, i.e. the condition which made the original approach useless is necessary for our new approach. This second order Equation leads to no amplitude equation, the motion is marginal even to the first nonlinear order.

The third order Eq. (57) reduces to

$$\begin{aligned} \mathbf{F}_{00}(\boldsymbol{\varphi}_3) = \tau_2\,\varrho_{00}\,\boldsymbol{\Phi}_1 \ddot{A}_1 - \{&\mathbf{G}_{00}(\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_{20})\,a_{12} \\ &+ \mathbf{G}_{00}(\boldsymbol{\Phi}_{20}, \boldsymbol{\Phi}_1)\,a_{21} \\ &+ \tau_2\,\mathbf{F}_{01}(\boldsymbol{\Phi}_1) A_1 \\ &+ [\mathbf{G}_{00}(\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_{22}) \\ &+ 2\,\mathbf{G}_{00}(\boldsymbol{\Phi}_{22}, \boldsymbol{\Phi}_1) \\ &+ \mathbf{H}_{00}(\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_1, \boldsymbol{\Phi}_1)]\,A_1{}^3/6\}\,. \end{aligned} \tag{63}$$

The integrability condition for internal modes becomes

$$\begin{aligned} &\text{a)} \quad \ddot{A}_1 = -\gamma_T{}^2 A_1 + 2(\vartheta_T/\tau_2) A_1{}^3 \quad \text{for} \quad \lambda > \lambda_0\,, \\ &\text{b)} \quad \ddot{A}_1 = -\gamma_T{}^2 A_1 + 2(\vartheta_T/\tau_2) A_1{}^3 \quad \text{for} \quad \lambda < \lambda_0 \end{aligned} \tag{64}$$

which are again nonlinear amplitude Equations for the marginal mode.

There are important applications for which $\delta_T = 0$: if the family of equilibria considered has an ignorable coordinate $\theta$ ($\partial p_0/\partial\theta \equiv 0$ ect. ...) — e.g. the toroidal angle in tokamaks — and if $\boldsymbol{\Phi}_1 \sim \exp(i n \theta)$ — e.g. kinkmodes in tokamaks — then $\delta_T = 0$ since $\delta_T$ is of third order in $\boldsymbol{\Phi}_1$. Also, for $\partial\boldsymbol{\Phi}_1/\partial\theta = 0$ we may have $\delta_T = 0$, e.g. for axisymmetric modes in tokamaks which are antisymmetric with respect to the equatorial plane [15].

### *1. Case* $\delta_T \neq 0$

In this case, the nonlinear motion for linearly unstable modes is to lowest order determined by (42). If, using (16), we return from $T$ to $t$, we obtain

$$\mathrm{d}^2 A_1/\mathrm{d}t^2 = \tau\,\gamma_T{}^2 A_1 + (3/2)(\tau/\tau_1)\,\delta_T A_1{}^2\,. \tag{65}$$

Let us now introduce a total amplitude $A$ defined by

$$A = \varepsilon A_1 + \varepsilon^2 A_2 + \cdots\,. \tag{66}$$

According to Eq. (20) for $\boldsymbol{\varphi}$, Eqs. (37), (46) and (49), $A$ is the amplitude with which $\boldsymbol{\Phi}_1$ appears in $\boldsymbol{\varphi}$.

Inserting the expansions (66) and (10) into (65), we obtain to lowest order in $\varepsilon$

$$\mathrm{d}^2 A/\mathrm{d}t^2 = \tau\,\gamma_T{}^2 A + (3/2)\,\delta_T A^2\,. \tag{67}$$

For linearly stable modes, we obtain from (42a) this Equation with $\gamma_T{}^2$ replaced by $-\gamma_T{}^2$ and with $\tau$ defined by (3a). If for both $\lambda > \lambda_0$ and $\lambda < \lambda_0$ we uniformly use the definition (3) of $\tau$, we get (67) for both cases. From (67) we obtain after one time integration

$$\begin{aligned} (\mathrm{d}A/\mathrm{d}t)^2 + V(A) &= \text{const}\,, \\ V(A) &= -(\tau\,\gamma_T{}^2 + \delta_T A) A^2\,. \end{aligned} \tag{68}$$

According to our assumption (1), we have

$$\gamma_T{}^2 > 0\,. \tag{69}$$

Since both $\boldsymbol{\varphi}_1$ and $-\boldsymbol{\varphi}_1$ are solutions of Eq. (35) and since $\delta_T$ is of third order in $\boldsymbol{\Phi}_1$ (Eq. (44)), $\boldsymbol{\varphi}_1$ can always be chosen such that

$$\delta_T \geqq 0\,. \tag{70}$$

Figure 2a shows the "potential" $V(A)$ for $\lambda > \lambda_0$ and $\lambda < \lambda_0$, $|\lambda - \lambda_0|$ having the same value in both cases.

The time behaviour of $A(t)$ can be deduced from the analogy of (68) with the motion of a masspoint

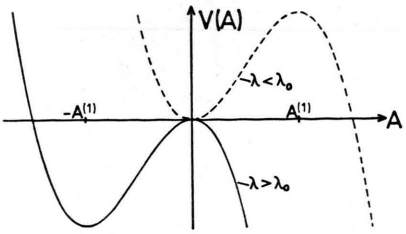


Fig. 2a. Potentials $V(A)$, Eq. (68), for $\delta_{\mathrm{T}} \neq 0$.

$$A^{(1)} = \frac{2\gamma_{\mathrm{T}}^2 |\lambda - \lambda_0|}{3\delta_{\mathrm{T}}}$$

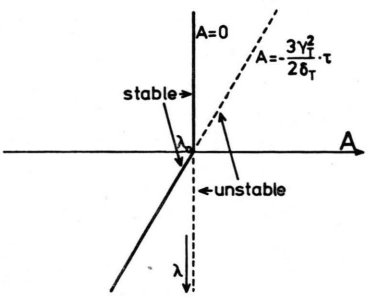


Fig. 2b. Bifurcation diagram for $\delta_T \neq 0$. Instead of $A = -(3\gamma_{\mathrm{T}}^2/2\delta_{\mathrm{r}})\,\tau$ it should read $A = -(2\gamma_{\mathrm{T}}^2/3\delta_{\mathrm{r}})\,\tau$.

in a potential $V$. Obviously, there exist new equilibrium positions at

$$A = -(2\,\gamma_{\mathrm{T}}^2/3\,\delta_{\mathrm{T}})\,\tau \tag{71}$$

which are shown in the bifurcation diagram Fig. 2b together with the set of unperturbed equilibria at $A = 0$. According to Fig. 2a, the bifurcating equilibria are obviously unstable for $\tau < 0$. For $\tau > 0$, they are obviously stable with respect to the motion $\boldsymbol{\varphi}_1$. Since we have assumed $\boldsymbol{\varphi}_1$ to be an isolated eigenmode, according to [16] they are generally linearly stable in the neighbourhood of $\lambda_0$.

As for the time dependent solutions of Eq. (68), let us first consider the case $\lambda > \lambda_0$: The solution

$$A = C \exp(\sqrt{\tau}\,\gamma_{\mathrm{T}}\,t) \tag{72}$$

of the linearized equation corresponds to the choice const $= 0$. For this choice, the nonlinear solution is

$$A = [(1+u)^2/(1-u)^2 - 1]\,\gamma_{\mathrm{T}}^2/\delta_{\mathrm{T}}\,,$$
$$u = C \exp(\sqrt{\tau}\,\gamma_{\mathrm{T}}\,t)\,, \tag{73}$$

where $C$ is a dimensionless integration constant. For $C < 0$ we get $A \leqq 0$ and the motion is a nonlinear oscillation of infinite period about the equilibrium (71). For $C > 0$ we have $A \geqq 0$ and get $A = \infty$ at the finite time $t = -(\ln C)/(\sqrt{\tau}\,\gamma_{\mathrm{T}})$: this is called explosive instability.

For const $< 0$, the motion is either an oscillation with $A < 0$ and finite oscillation period — frictional effects not contained in ideal MHD will then gradually damp out these oscillations and drive the plasma into the equilibrium (71) — or, we get again an instability of the explosive type with $A > 0$.

For const $> 0$, the motion is generally an explosive instability. It may go down the slope at $A \geqq 0$ either immediately or subsequently to one oscillation to the side $A \leqq 0$.

While the linear solution (72) is symmetric with respect to the sign of $A$, there exists a remarkable asymmetry in the nonlinear solution. An example will illustrate how this asymmetry comes about. Let us consider axisymmetric modes in tokamaks which are symmetric with respect to the equatorial plane. This is just the situation for which $\delta_{\mathrm{T}} \neq 0$. According to [17] there appears a linear instability of this type for elliptical plasma cross-sections, if the vertical elongation of the ellipse becomes too large. For $A > 0$ or $A < 0$ respectively this instability increases or respectively decreases the elongation, i.e. $A \gtrless 0$ represent physically asymmetric situations. Increasing the elongation reinforces the cause of instability, which makes an explosive enhancement quite plausible. Decreasing the elongation weakens the cause and would explain the occurence of an oscillation.

Let us now consider the case $\lambda < \lambda_0$: According to Fig. 2a the motion will remain a nonlinear oscillation about $A = 0$ as long as the initial perturbation is weak enough. However, if this one is so strong that $A$ can get beyond the unstable equilibrium position (71), we get again an explosive instability, which might be preceded by one oscillation towards $A \leqq 0$. Thus, the plasma may be nonlinearly unstable although it is stable according to linear stability theory, i.e. the stability boundary is shifted into the linearly stable regime. According to (3) and (71), the nonlinear stability boundary is given by

$$\lambda = \lambda_0 - (3\,\delta_{\mathrm{T}}/2\,\gamma_{\mathrm{T}}^2)\,A\,, \tag{74}$$

where $A$ is the minimum necessary amplitude for instability. Amplitude dependence of the stability boundary is a well known phenomenon in nonlinear stability.

We briefly comment on what may be expected from higher order corrections. Obviously, the plasma motion cannot be described correctly by an explosive instability until $A = \infty$, since this would contradict energy conservation. (The same argument already holds for exponential instability.) Thus, after some time, higher order corrections must become important. The next order Eq. (58) has the structure

$$\ddot{A}_2 = \gamma^2(T) A_2 + f(T),$$
$$\gamma^2 = \gamma_{\mathrm{T}}^2 + 3(\delta_{\mathrm{T}}/\tau_1) A_1(T), \tag{75}$$

where $f(T)$ depends on $T$ through $A_1(T)$. This is a linear equation for $A_2$ with given forcing term $f(T)$ and given time dependent growth rate $\gamma(T)$ since $A_1(T)$ is known from the lower order. For a nonlinear oscillation $A_1(T)$, $f(T)$ and $\gamma(T)$ are periodic functions, and (75) is a Hill's Equation with a periodic forcing term. Parametric instability may occur in this case, i.e. the nonlinear stabilization of the linear instability caused by the periodicity of $A_1$ may be destroyed in the next order. If $A_1(T)$ is explosive, due to the timedependence of $\gamma(T)$ and due to the forcing term, $A_2$ may increase even more dramatically. However, it may counteract the increase of $A_1$ and thus weaken the explosive decay.

A more detailed discussion of (58) will be presented in a following paper.

*2. Case* $\delta_{\mathrm{T}} = 0$

In this case we have to deal with Eqs. (64). As in the case $\delta_{\mathrm{T}} \neq 0$, using (16) we return from $T$ to $t$

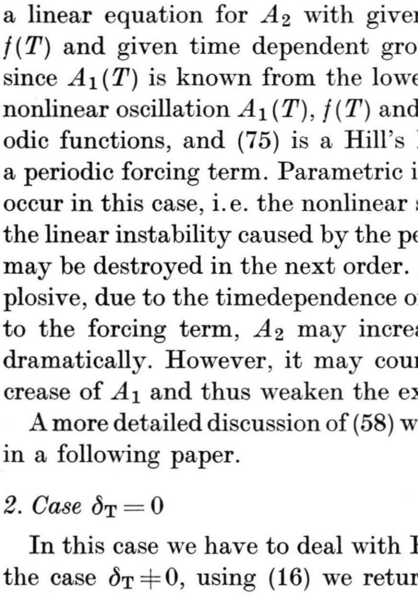


Fig. 3a. Potentials $V(A)$, Eq. (77), for $\delta_{\mathrm{T}} = 0$, $\vartheta_{\mathrm{T}} > 0$ $A^{(2)} = \gamma_{\mathrm{T}} \left(\frac{|\lambda - \lambda_0|}{2\vartheta_{\mathrm{T}}}\right)^{1/2}$.

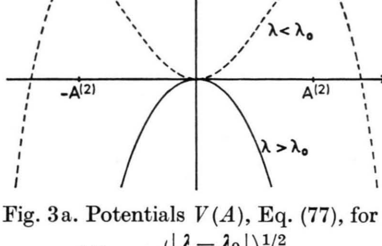


Fig. 3 b. Bifurcation diagram for $\delta_{\mathrm{T}} = 0$, $\vartheta_{\mathrm{T}} > 0$. Instead $A = \pm \gamma_{\mathrm{T}} \sqrt{-\tau/\vartheta_{\mathrm{T}}}$ it should read $A = \pm \gamma_{\mathrm{T}} \sqrt{-\tau/(2)}$.

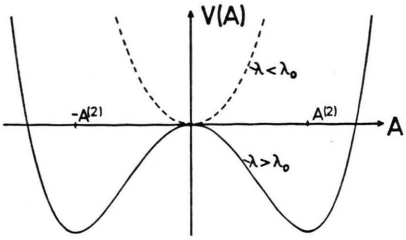


Fig. 4a. Potentials $V(A)$, Eq. (77), for $\delta_T = 0$, $\vartheta_T < 0$ $A^{(2)} = \gamma_{\mathrm{T}} \left(\frac{|\lambda - \lambda_0|}{2\vartheta_{\mathrm{T}}}\right)^{1/2}$.

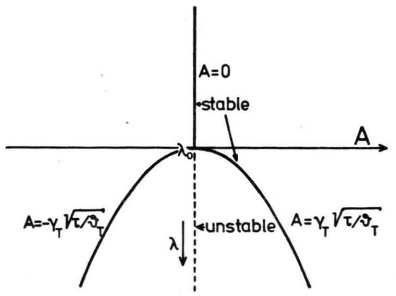


Fig. 4b. Bifurcation diagram for $\delta_{\mathrm{T}} = 0$, $\vartheta_{\mathrm{T}} < 0$. Instead of $A = \pm \gamma_{\mathrm{T}} \sqrt{\tau/\vartheta_{\mathrm{T}}}$ it should read $A = \pm \gamma_{\mathrm{T}} \sqrt{-\tau/(2\vartheta_{\mathrm{T}})}$.

and make use of the expansions (10) and (66). Using uniquely the definition (3) of $\tau$, to lowest order in $\varepsilon$ we obtain both for $\lambda < \lambda_0$ and $\lambda > \lambda_0$

$$\mathrm{d}^2 A/\mathrm{d}t^2 = \tau \gamma_{\mathrm{T}}^2 A + 2\vartheta_{\mathrm{T}} A^3 \tag{76}$$

and from this after one time integration (see [20], p.829)

$$(\mathrm{d}A/\mathrm{d}t)^2 + V(A) = \text{const}, \tag{77}$$
$$V(A) = -(\tau \gamma_{\mathrm{T}}^2 + \vartheta_{\mathrm{T}} A^2) A^2.$$

A linearly stable ($\lambda < \lambda_0$) and a linearly unstable ($\lambda > \lambda_0$) potential $V(A)$ is shown for $\vartheta_{\mathrm{T}} > 0$ in Fig. 3a) and for $\vartheta_{\mathrm{T}} < 0$ in Figure 4a). In both cases, $|\lambda - \lambda_0|$ has the same value for $\lambda < \lambda_0$ and $\lambda > \lambda_0$. Besides $A = 0$, from (76) we obtain also equilibrium positions for

$$A^2 = -(\gamma_{\mathrm{T}}^2/2\vartheta_{\mathrm{T}})\tau. \tag{78}$$

If $\vartheta_{\mathrm{T}} > 0$, this Equation has a solution only for $\tau < 0$. If $\vartheta_{\mathrm{T}} < 0$, solutions exist only for $\tau > 0$. The bifurcating equilibria are shown in the bifurcation diagrams 3b) and 4b) respectively. In both cases we have parabolic bifurcation. According to Fig. 3a the bifurcating equilibria of Fig. 3b are unstable, while according to Fig. 4a and [16], the bifurcating equilibria of Fig. 4b are stable.

Turning to the nonlinear motions, we first consider the case $\vartheta_{\mathrm{T}} > 0$. For $\lambda > \lambda_0$, all motions are explosive instabilities, which for the choice const = 0 in (77) are given by

$$A = 2u/[1 - (\vartheta_T/\gamma_T^2)u^2], \quad u = C\exp(\sqrt{\tau}\gamma_T t). \tag{79}$$

For $\lambda < \lambda_0$, the motions are nonlinear oscillations about $A = 0$ if the perturbation is weak enough. If the perturbation amplitude $A$ can get beyond the unstable equilibrium positions of (78), we obtain an explosive instability again. The stability boundary is nonlinearly shifted towards

$$\lambda = \lambda_0 - 2(\vartheta_T/\gamma_T^2)A^2. \tag{80}$$

For $\vartheta_T < 0$, the motion is a nonlinear oscillation either about $A = 0$, if $\lambda < \lambda_0$, or, if $\lambda > \lambda_0$, about one of the two stable equilibria represented by (78).

*Induced Instabilities*

In both cases $\delta_T \neq 0$ and $\delta_T = 0$, additional modes are induced to the first nonlinear order according to (45)—(46). These, for $\lambda > \lambda_0$, are induced instabilities. The modes $\boldsymbol{\varphi}_{21}$ and $\boldsymbol{\varphi}_{22}$ are "slaved" [18] by the lowest order instability $\boldsymbol{\varphi}_1$ since their time evolution is completely determined by the one of $\boldsymbol{\varphi}_1$ through their dependence on $A_1(T)$.

The induced modes do not play any role in the amplitude Eq. (42) for the case $\delta_T \neq 0$ since the coefficient $\delta_T$ of the nonlinear term only depends on $\boldsymbol{\varphi}_1$. Quite in contrast, through $\vartheta_T$ (see Eqs. (59)) the nonlinear motion (76) depends on the induced mode $\boldsymbol{\varphi}_{22}$ as well as on $\boldsymbol{\varphi}_1$ for the case $\delta_T = 0$. Although $\boldsymbol{\varphi}_2$ is a lower order contribution as compared to $\boldsymbol{\varphi}_1$, this is possible since according to (46) $\boldsymbol{\varphi}_{22}$ grows much faster than $\boldsymbol{\varphi}_1$ so that

$$|\varepsilon\boldsymbol{\varphi}_{22}|/|\boldsymbol{\varphi}_1| \sim 1$$

after a finite time.

## Summary and Conclusions

Nonlinear amplitude equations for the linearly marginal mode were derived. These are conditions of integrability for higher order equations obtained in a reductive perturbation analysis. Depending on the symmetry of the problem, two different cases are possible: For the one case, an explosive instability generally appears in the linearly unstable regime, while for the other, either an explosive instability or a nonlinear oscillation is possible. Whenever a linear instability is nonlinearly turned into an explosive instability, the stability limit is shifted into the linearly stable regime. Additional modes are induced in the first nonlinear order which, however, affect the lowest nonlinear amplitude equation only in one of both cases.

There is a close relation between the type of nonlinear motion and the bifurcation of dynamically connected equilibria at $\lambda_0$, which is observed in all cases. Bifurcation of equilibria might be restricted to the case of eigenmodes with isolated eigenvalues, which we considered in this paper. The bifurcating equilibria obtained satisfy the equilibrium equations only approximately, and there must not exist exact equilibria which are approximated by these. In tokamaks for example, the appearence of the kinkmode is accompanied by bifurcation of stellarator-like equilibria carrying toroidal current. The existence of such equilibria is not proven and is even doubtful [19].

The analysis of this paper is restricted to the case of internal modes. For external modes, the nonlinear boundary conditions on the plasma-vacuum interface must be treated. The same analysis is possible, and the same amplitude equations are obtained, the determination of coefficients being still more complicated [14]. Specific applications of the theory presented and higher order effects which may lead to a saturation of explosive instabilities will be treated in following papers.

*Acknowledgements*

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## Appendix

a) $\boldsymbol{j}, \boldsymbol{B}, \boldsymbol{v}, \boldsymbol{\varphi}, \boldsymbol{\Phi}$, etc. and $\varrho$, $p$ resp., are vector- and scalar fields resp., $\mathbf{B}_1$, $\mathbf{B}_{10}$, $\mathbf{F}_0$, $\mathbf{F}_{00}$, $\mathbf{G}_{00}$ etc. are vector operators, while $\not{p}_1$, $\not{p}_{10}$ etc are scalar operators.

b) List of operators

$$\dot{f} = \mathrm{d}f/\mathrm{d}T, \tag{A1}$$

$$(\boldsymbol{\varphi}, \boldsymbol{\chi}) = \int_{Pl} \boldsymbol{\varphi}\cdot\boldsymbol{\chi}\,\mathrm{d}\tau, \tag{A2}$$

where the integral extends over the whole plasma region.

$$\mathbf{B}_1(\boldsymbol{\varphi},\boldsymbol{\chi}) = \nabla\times(\boldsymbol{\varphi}\times\boldsymbol{\chi}), \tag{A3}$$

$$p_1(\boldsymbol{\varphi},\chi) = -[\boldsymbol{\varphi}\cdot\nabla\chi + (5/3)\chi\nabla\cdot\boldsymbol{\varphi}], \tag{A4}$$

$$\mathbf{B}_{1i}(\boldsymbol{\varphi}) = \mathbf{B}_1(\boldsymbol{\varphi}, \boldsymbol{B}_{0i}), \quad i = 0,1,2, \tag{A5}$$

$$p_{1i}(\boldsymbol{\varphi}) = p_1(\boldsymbol{\varphi}, p_{0i}), \quad i = 0,1,2, \tag{A6}$$

$$\mathbf{F}_0(\boldsymbol{\xi}) = \boldsymbol{j}_0\times\mathbf{B}_1(\boldsymbol{\xi},\boldsymbol{B}_0) + [\boldsymbol{\nabla}\times\mathbf{B}_1(\boldsymbol{\xi},\boldsymbol{B}_0)]\times\boldsymbol{B}_0 - \boldsymbol{\nabla}p_1(\boldsymbol{\xi},p_0), \tag{A7}$$

$$\mathbf{F}_{00}(\boldsymbol{\varphi}) = \boldsymbol{j}_{00}\times\mathbf{B}_{10}(\boldsymbol{\varphi}) + [\boldsymbol{\nabla}\times\mathbf{B}_{10}(\boldsymbol{\varphi})]\times\boldsymbol{B}_{00} - \boldsymbol{\nabla}p_{10}(\boldsymbol{\varphi}), \tag{A8}$$

$$\begin{aligned}\mathbf{F}_{01}(\boldsymbol{\varphi}) = {} & \boldsymbol{j}_{00}\times\mathbf{B}_{11}(\boldsymbol{\varphi}) + [\boldsymbol{\nabla}\times\mathbf{B}_{11}(\boldsymbol{\varphi})]\times\boldsymbol{B}_{00} - \boldsymbol{\nabla}p_{11}(\boldsymbol{\varphi}) \\ & + \boldsymbol{j}_{01}\times\mathbf{B}_{10}(\boldsymbol{\varphi}) + [\boldsymbol{\nabla}\times\mathbf{B}_{10}(\boldsymbol{\varphi})]\times\boldsymbol{B}_{01},\end{aligned} \tag{A9}$$

$$\begin{aligned}\mathbf{F}_{02}(\boldsymbol{\varphi}) = {} & \boldsymbol{j}_{00}\times\mathbf{B}_{12}(\boldsymbol{\varphi}) + [\boldsymbol{\nabla}\times\mathbf{B}_{12}(\boldsymbol{\varphi})]\times\boldsymbol{B}_{00} - \nabla p_{12}(\boldsymbol{\varphi}) + \boldsymbol{j}_{02}\times\mathbf{B}_{10}(\boldsymbol{\varphi}) \\ & + [\nabla\times\mathbf{B}_{10}(\boldsymbol{\varphi})]\times\boldsymbol{B}_{02} + \boldsymbol{j}_{01}\times\mathbf{B}_{11}(\boldsymbol{\varphi}) + [\nabla\times\mathbf{B}_{11}(\boldsymbol{\varphi})]\times\boldsymbol{B}_{01},\end{aligned} \tag{A10}$$

$$\begin{aligned}\mathbf{G}_{00}(\boldsymbol{\varphi},\boldsymbol{\chi}) = {} & \boldsymbol{j}_{00}\times\mathbf{B}_1(\boldsymbol{\varphi},\mathbf{B}_{10}(\boldsymbol{\chi})) + [\boldsymbol{\nabla}\times\mathbf{B}_1(\boldsymbol{\varphi},\mathbf{B}_{10}(\boldsymbol{\chi}))]\times\boldsymbol{B}_{00} - \nabla p_1(\boldsymbol{\varphi},p_{10}(\boldsymbol{\chi})) \\ & + [\nabla\times\mathbf{B}_{10}(\boldsymbol{\varphi})]\times\mathbf{B}_{10}(\boldsymbol{\chi}) + [\nabla\times\mathbf{B}_{10}(\boldsymbol{\chi})]\times\mathbf{B}_{10}(\boldsymbol{\varphi}),\end{aligned} \tag{A11}$$

$$\begin{aligned}\mathbf{G}_{01}(\boldsymbol{\varphi},\boldsymbol{\varphi}) = {} & \boldsymbol{j}_{00}\times\mathbf{B}_1(\boldsymbol{\varphi},\mathbf{B}_{11}(\boldsymbol{\varphi})) + [\nabla\times\mathbf{B}_1(\boldsymbol{\varphi},\mathbf{B}_{11}(\boldsymbol{\varphi}))]\times\boldsymbol{B}_{00} - \nabla p_1(\boldsymbol{\varphi},p_{11}(\boldsymbol{\varphi})) \\ & + \boldsymbol{j}_{01}\times\mathbf{B}_1(\boldsymbol{\varphi},\mathbf{B}_{10}(\boldsymbol{\varphi})) + [\nabla\times\mathbf{B}_1(\boldsymbol{\varphi},\mathbf{B}_{10}(\boldsymbol{\varphi}))\times\boldsymbol{B}_{01} \\ & + 2[\nabla\times\mathbf{B}_{10}(\boldsymbol{\varphi})]\times\mathbf{B}_{11}(\boldsymbol{\varphi}) + 2[\nabla\times\mathbf{B}_{11}(\boldsymbol{\varphi})]\times\mathbf{B}_{10}(\boldsymbol{\varphi}),\end{aligned} \tag{A12}$$

$$\begin{aligned}\mathbf{H}_{00}(\boldsymbol{\varphi},\boldsymbol{\varphi},\boldsymbol{\varphi}) = {} & \boldsymbol{j}_{00}\times\mathbf{B}_1(\boldsymbol{\varphi},\mathbf{B}_1(\boldsymbol{\varphi},\mathbf{B}_{10}(\boldsymbol{\varphi}))) + [\nabla\times\mathbf{B}_1(\boldsymbol{\varphi},\mathbf{B}_1(\boldsymbol{\varphi},\mathbf{B}_{10}(\boldsymbol{\varphi})))]\times\boldsymbol{B}_{00} \\ & - \nabla p_1(\boldsymbol{\varphi},p_1(\boldsymbol{\varphi},p_{10}(\boldsymbol{\varphi}))) \\ & + 3[\boldsymbol{\nabla}\times\mathbf{B}_{10}(\boldsymbol{\varphi})]\times\mathbf{B}_1(\boldsymbol{\varphi},\mathbf{B}_{10}(\boldsymbol{\varphi})) + 3[\boldsymbol{\nabla}\times\mathbf{B}_1(\boldsymbol{\varphi},\mathbf{B}_{10}(\boldsymbol{\varphi}))]\times\mathbf{B}_{10}(\boldsymbol{\varphi}).\end{aligned} \tag{A13}$$

c) Properties of the operators listed:

The linear operators (A5)—(A10) have the properties

$$\mathscr{L}(a\boldsymbol{\varphi} + b\boldsymbol{\chi}) = a\mathscr{L}(\boldsymbol{\varphi}) + b\mathscr{L}(\boldsymbol{\chi}), \tag{A14}$$

where

$$a = a(T), \quad b = b(T),$$

and

$$\partial\mathscr{L}(\boldsymbol{\varphi})/\partial T = \mathscr{L}(\partial\boldsymbol{\varphi}/\partial T). \tag{A15}$$

The bilinear operators (A3), (A4), (A11), and (A12) have the property

$$\begin{aligned}&\mathscr{G}(a_1\boldsymbol{\varphi}_1 + a_2\boldsymbol{\varphi}_2, b_1\boldsymbol{\chi}_1 + b_2\boldsymbol{\chi}_2) \\ &= a_1 b_1\mathscr{G}(\boldsymbol{\varphi}_1,\boldsymbol{\chi}_1) + a_2 b_1\mathscr{G}(\boldsymbol{\varphi}_2,\boldsymbol{\chi}_1) \\ &\quad + a_1 b_2\mathscr{G}(\boldsymbol{\varphi}_1,\boldsymbol{\chi}_2) + a_2 b_2\mathscr{G}(\boldsymbol{\varphi}_2,\boldsymbol{\chi}_2),\end{aligned} \tag{A16}$$

where for (A4) $\boldsymbol{\chi}_1$ and $\boldsymbol{\chi}_2$ must be replaced by scalars $\chi_1$ and $\chi_2$. $a_1$, $a_2$, $b_1$ and $b_2$ may be functions of $T$, $\mathscr{G}$ may be either a scalar or a vector operator.

The trilinear operators (A13), $\mathbf{B}_1(\boldsymbol{\varphi},\mathbf{B}_1(\boldsymbol{\varphi},\mathbf{B}_{10}(\boldsymbol{\varphi})))$ and $p_1(\boldsymbol{\varphi},p_1(\boldsymbol{\varphi},p_{10}(\boldsymbol{\varphi})))$ have the property

$$\mathscr{H}(a\boldsymbol{\varphi},a\boldsymbol{\varphi},a\boldsymbol{\varphi}) = a^3\mathscr{H}(\boldsymbol{\varphi},\boldsymbol{\varphi},\boldsymbol{\varphi}). \tag{A17}$$


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[20] *Note added in proof:*
After submission of this paper the author got knowledge of a paper by P.N. Hu, Phys. Fluids **23,** 337 (1980) where Eq. (77) of this paper, which is valid for some pertubation in tokamaks, is derived, while Eq. (68) of this paper valid for stellarators is missing. Hu's approach is different from ours in that the expansion is made with respect to the pertubation amplitude while the driving parameters are fixed and the operator left unexpanded. For practical applications, this method has the disadvantage that it starts with a nontrivial eigenmode problem to lowest order. On the other hand, the paper studies the influence of dissipative effects on nonlinear MHD modes.